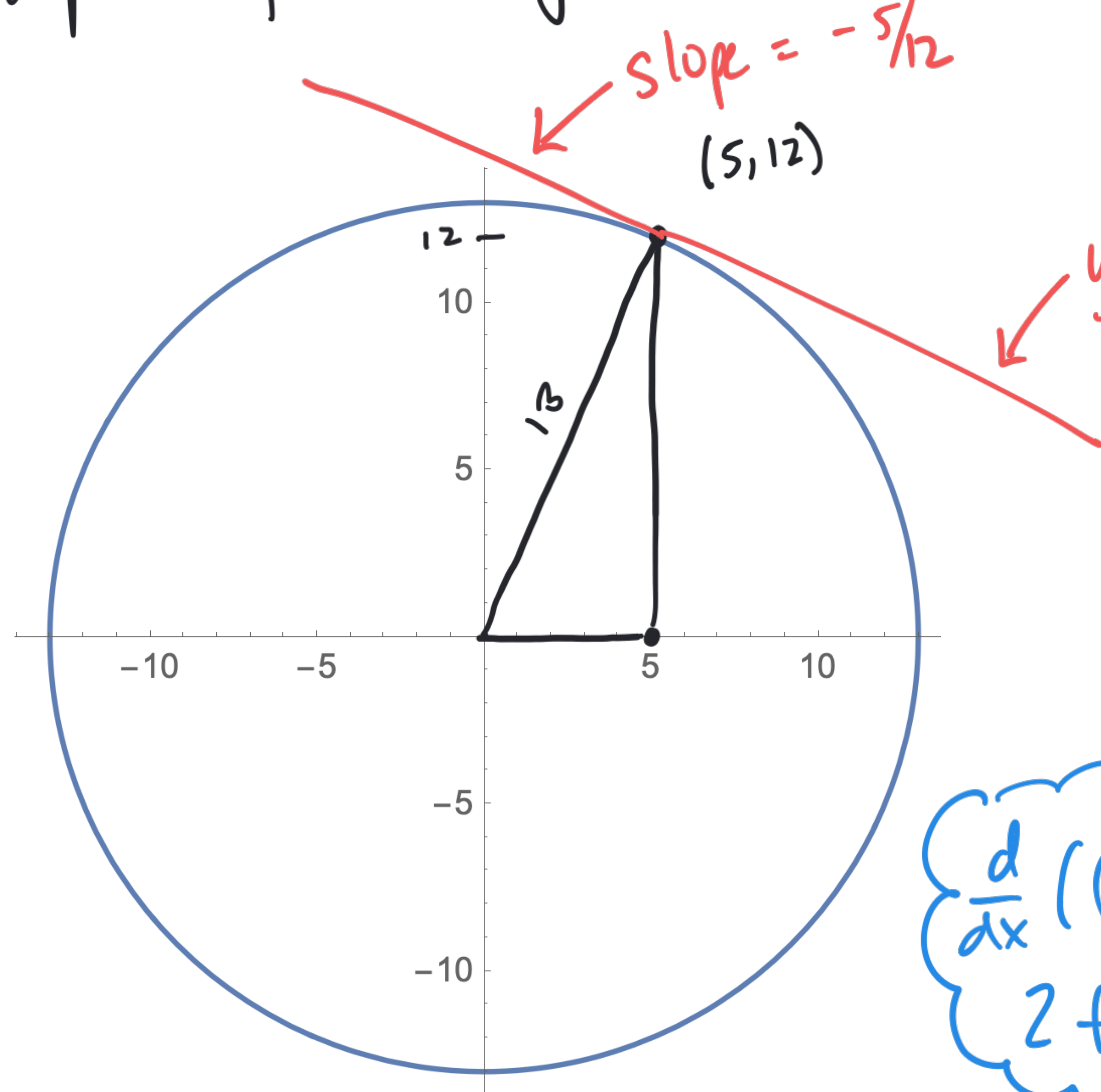
Intro Video: Section 3.5
Implicit Differentiation

Math F251X: Calculus I

Suppose you have a curve described in terms of x and y (not a function). Can we find slopes of tangent lines?



what is this slope?

$$x^2+y^2=169=13^2$$

y is some function of x

$$x^2+y^2=13^2$$

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}\left(13^2\right)$$

$$\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}\left(13^2\right)$$

$$2x+\frac{d}{dx}\left(y^2\right)=0$$

$$\frac{d}{dx}\left(\left(f(x)\right)^2\right)=2f(x)f'(x)$$

$$2x+2yy'=0$$

$$y'=\frac{-2x}{2y}=\frac{-x}{y}$$

$$y'\big|_{(5,12)}=\frac{-5}{12}$$

← x-value of point
← y value of point

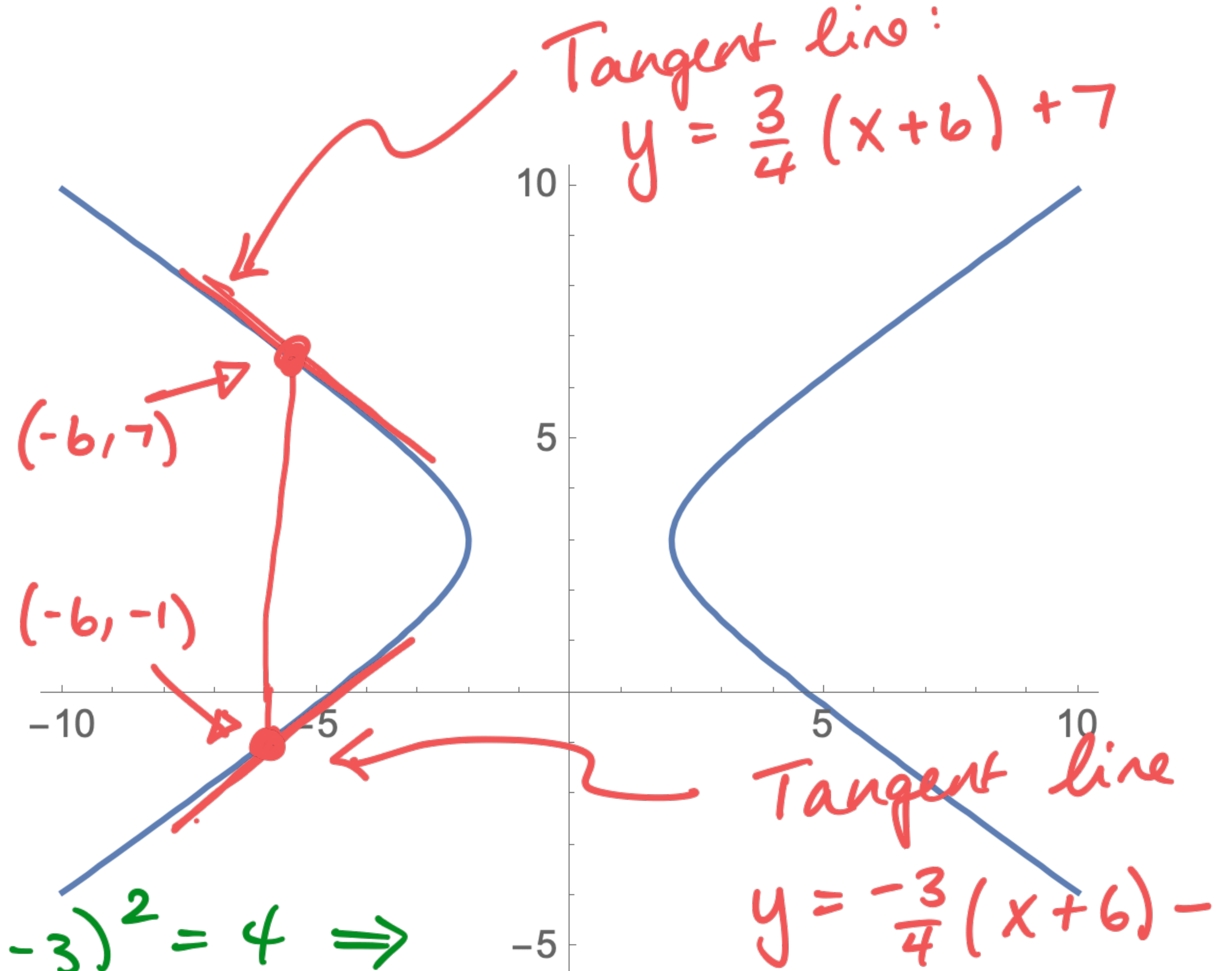
<u>Example</u>: Find the equation of the tangent line(s) to the curve $x^2 - 2(y-3)^2 = 4$ at the point(s) where $x = -6$.

$$\frac{d}{dx}\left(x^2 - 2(y-3)^2\right) = \frac{d}{dx}(4) \Rightarrow$$

$$\frac{d}{dx}(x^2) - 2\,\frac{d}{dx}\left((y-3)^2\right) = 0 \Rightarrow$$

$$2x - 2\cdot 2(y-3)\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = \frac{2x}{-4(y-3)} = \frac{x}{-2(y-3)}$$



What is y when $x = -6$?

Substitute $x = -6$ into $x^2 - 2(y-3)^2 = 4 \Rightarrow$

$(-6)^2 - 2(y-3)^2 = 4 \Rightarrow -2(y-3)^2 = 4 - 36 = -32 \Rightarrow (y-3)^2 = +16$

$\Rightarrow y - 3 = 4$ or $y - 3 = -4 \Rightarrow \boxed{y = 7 \text{ or } y = -1}$

$$\left.\frac{dy}{dx}\right|_{(-6,7)} = \frac{-6}{-2(7-3)} = \frac{-6}{-8} = \frac{3}{4}$$

$$\left.\frac{dy}{dx}\right|_{(-6,-1)} = \frac{-6}{-2(-1-3)} = \frac{-6}{8} = \frac{-3}{4}$$

<u>Example</u>: Find $\frac{dy}{dx}$ if $x^2 - xy + y^2 = 3$.

$$\frac{d}{dx}\left(x^2 - xy + y^2\right) = \frac{d}{dx}(3) \Rightarrow$$

$$2x - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0 \Rightarrow$$

$$2x - \left(x\frac{dy}{dx} + y(1)\right) + 2y\frac{dy}{dx} = 0 \Rightarrow$$

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0 \Rightarrow$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x \Rightarrow$$

$$\frac{dy}{dx}(2y - x) = y - 2x \Rightarrow$$

So $\boxed{\frac{dy}{dx} = \frac{y - 2x}{2y - x}}$

Example Find $\frac{dy}{dx}$ if $\cos(xy) = 1 + \sin(y)$.

$$\frac{d}{dx}\left(\cos(xy)\right) = \frac{d}{dx}\left(1 + \sin(y)\right)$$

$$-\sin(xy)\left[x\frac{dy}{dx} + y(1)\right] = \cos(y)\frac{dy}{dx}$$

$$-x\sin(xy)\frac{dy}{dx} - y\sin(xy) = \frac{dy}{dx}\left(\cos(y)\right) \Rightarrow$$

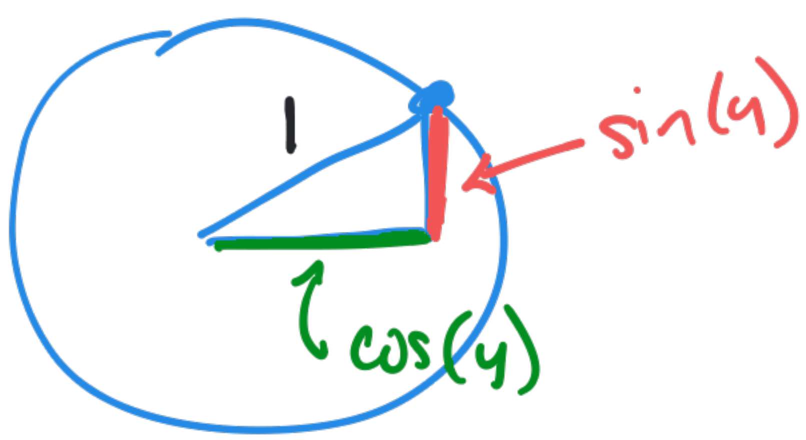
$$\frac{dy}{dx}\left(\cos(y) + x\sin(xy)\right) = -y\sin(xy) \Rightarrow$$

Derivatives of Inverse Trig functions:

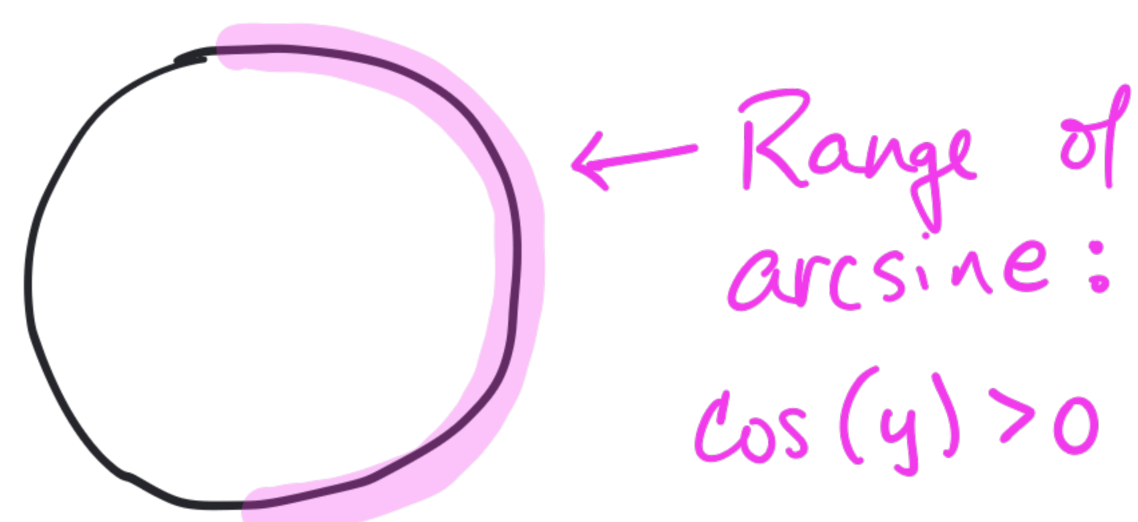
What is the derivative of $y = \arcsin(x)$?

$$y = \arcsin(x) \iff x = \sin(y).$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin(y)) \Rightarrow 1 = \cos(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$$



$$1 = (\sin(y))^2 + (\cos(y))^2 \Rightarrow (\cos(y))^2 = 1 - (\sin(y))^2$$



$$\cos(y) = \sqrt{1 - (\sin(y))^2}$$
$$= \sqrt{1 - x^2}$$

Therefore, $\frac{d}{dx}(y) = \frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$.

Example: Find the point(s) on the curve $x^3+y^3=6xy$ where the tangent line is horizontal.

$\rightarrow \frac{dy}{dx}=0$ for what (x,y)?

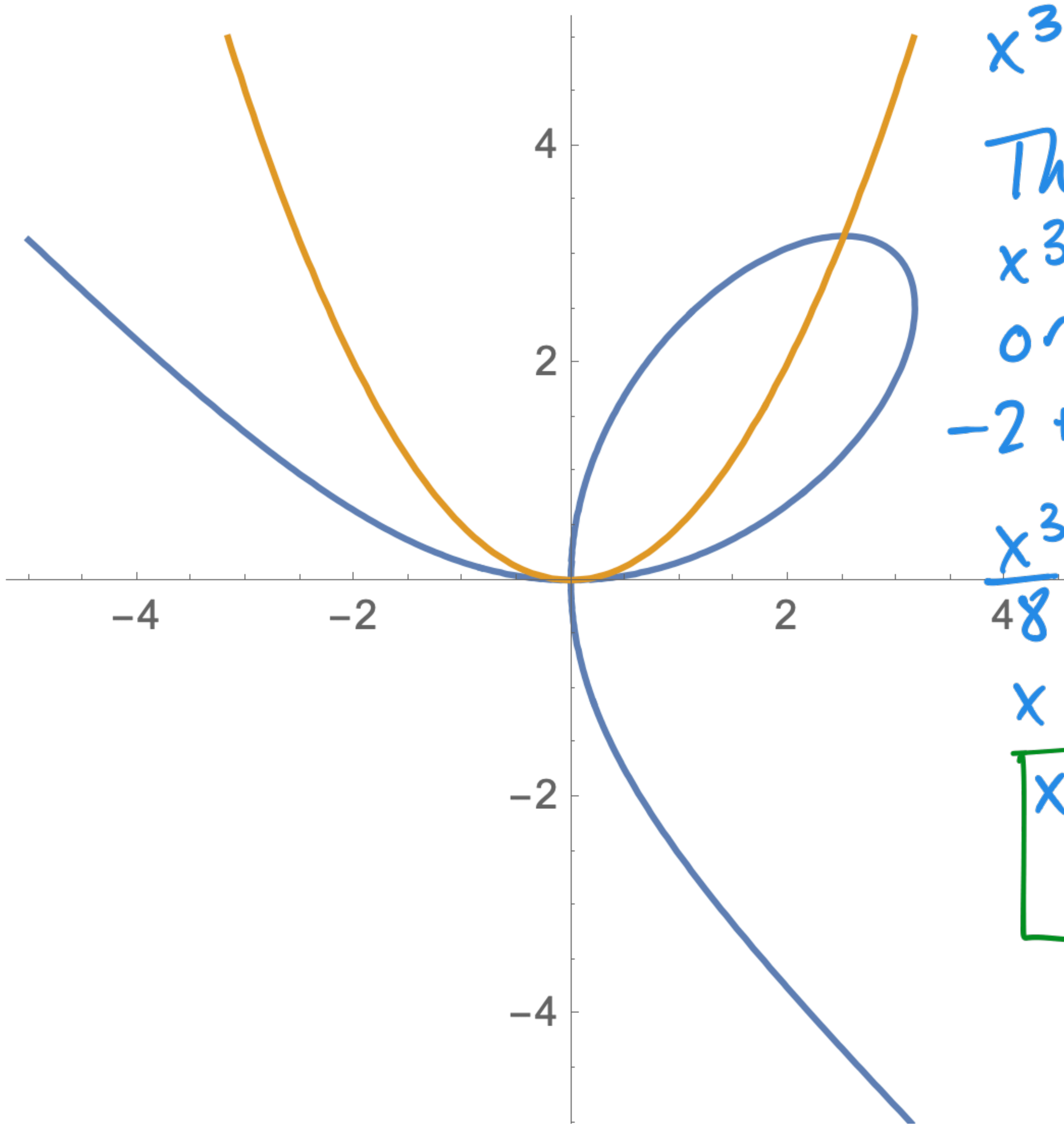
$$\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(6xy)$$

$$3x^2+3y^2\frac{dy}{dx}=6\left(x\frac{dy}{dx}+y(1)\right)$$

$$3x^2+3y^2\frac{dy}{dx}=6x\frac{dy}{dx}+6y \Rightarrow$$

$$3y^2\frac{dy}{dx}-6x\frac{dy}{dx}=6y-3x^2 \Rightarrow$$

$$\frac{dy}{dx}=\frac{6y-3x^2}{3y^2-6x}=\frac{2y-x^2}{y^2-2x}$$



So:
$x^3\left(-2+\frac{x^3}{8}\right)=0$

Thus:
$x^3=0 \Rightarrow \boxed{x=0}$

or

$-2+\frac{x^3}{8}=0 \Rightarrow$

$\frac{x^3}{8}=2 \Rightarrow$

$x^3=16 \Rightarrow$

$\boxed{x=\sqrt[3]{16}=2\sqrt[3]{2}}$

So $\frac{dy}{dx}=0 \Rightarrow \frac{6y-3x^2}{3y^2-6x}=0 \Rightarrow 6y-3x^2=0 \Rightarrow 3x^2=6y \Rightarrow y=\frac{x^2}{2}$.

Need

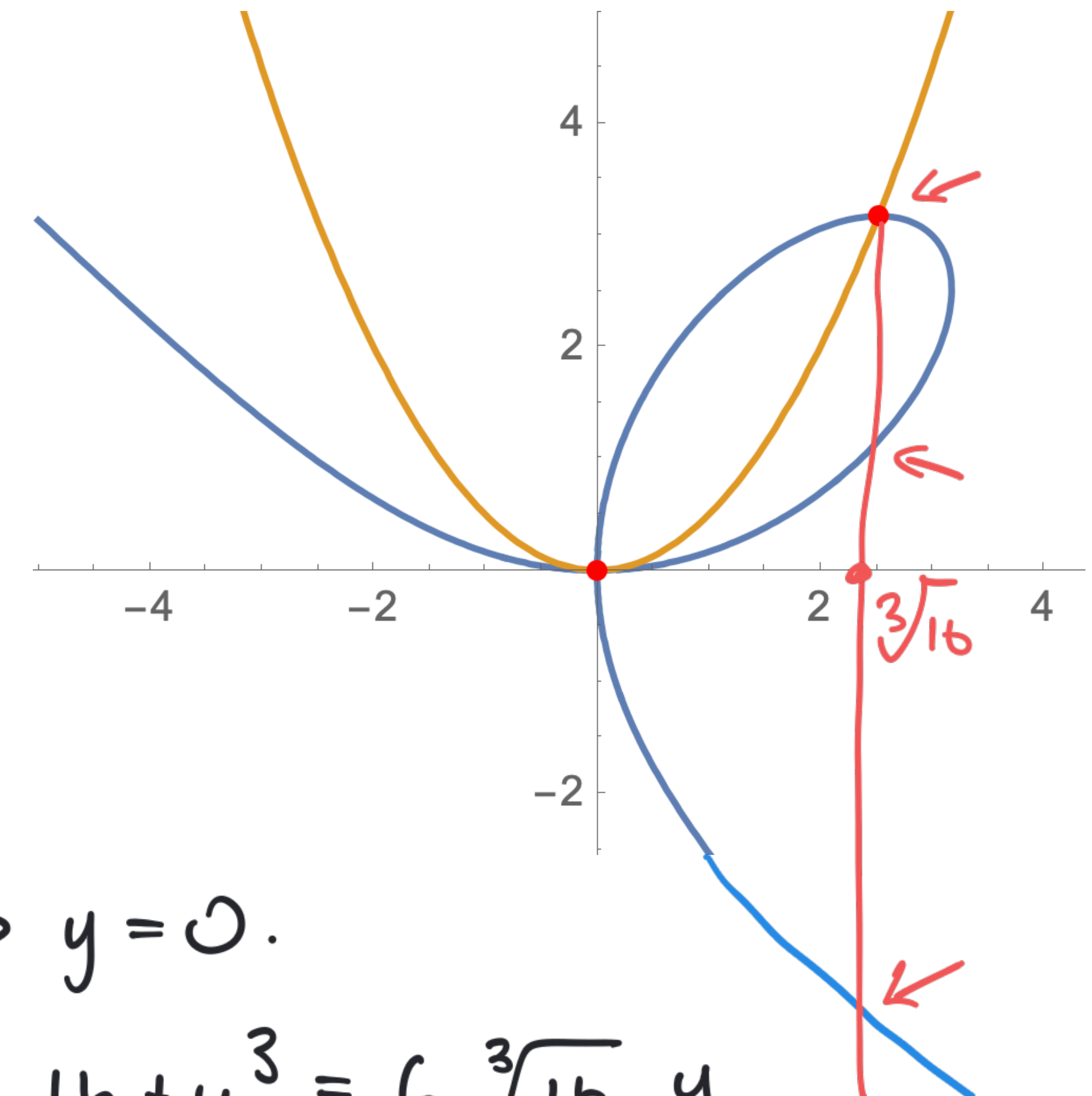
$$x^3+\left(\frac{x^2}{2}\right)^3=6x\left(\frac{x^2}{2}\right) \Rightarrow x^3+\frac{x^6}{8}=3x^3 \Rightarrow x^3-3x^3+\frac{x^6}{8}=0 \Rightarrow x^3\left(-2+\frac{x^3}{8}\right)=0$$

Recap: ① Our curve is $x^3+y^3=6xy$

② $\dfrac{dy}{dx}=\dfrac{2y-x^2}{y^2-2x}$

③ for $\dfrac{dy}{dx}=0$, need $\boxed{y=\dfrac{x^2}{2}.}$

④ Substituting $y=\dfrac{x^2}{2}$ into the curve yields $x=0$ or $x=\sqrt[3]{16}=2\sqrt[3]{2}$



Substitute these into our curve:

$x=0$: $\quad 0^3+y^3=6(0)y \Rightarrow y^3=0 \Rightarrow y=0.$

$x=\sqrt[3]{16}$: $\quad (\sqrt[3]{16})^3+y^3=6(\sqrt[3]{16})y \Rightarrow 16+y^3=6\sqrt[3]{16}\,y$

$\Rightarrow y^3-6\sqrt[3]{16}\,y+16=0$

STOP! Substitute it into the other function, which controls when $\dfrac{dy}{dx}=0$. When $x=\sqrt[3]{16}$, $y=\dfrac{(\sqrt[3]{16})^2}{2}=16^{2/3}\cdot 2^{-1}=(2^4)^{2/3}(2^{-1})=2^{5/3}$
$=2\sqrt[3]{2^2}$

ANSWER: TL is horizontal at $(0,0)$ and $(2\sqrt[3]{2},\ 2\sqrt[3]{2^2})$